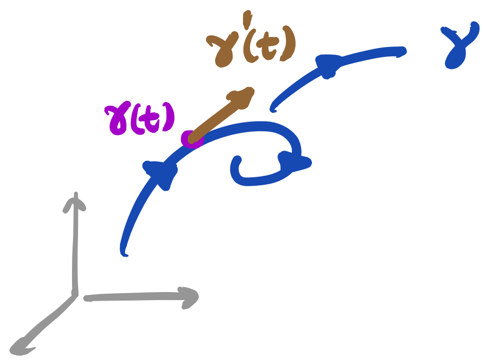
MATH 2028 - Line Integrals

GOAL: Define the line integral of functions and vector fields along curves in $\mathbb{R}^n$

First, we recall:

Def$^{\underline{n}}$: A regular parametrized curve in $\mathbb{R}^n$ is a C^1 map $\gamma : [a,b] \to \mathbb{R}^n$ s.t.

$$\gamma'(t) \neq \vec{0} \qquad \forall\, t \in [a,b]$$



Written in coordinates:

C^1 functions of t

$$\gamma(t) = (\gamma_1(t), \dots, \gamma_n(t))$$

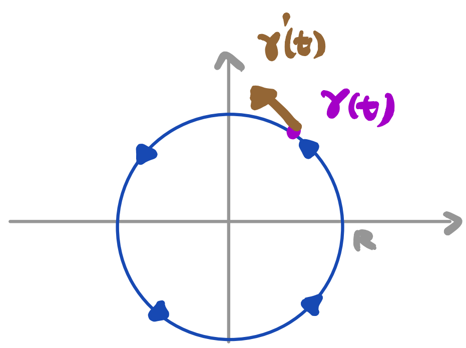
$$\gamma'(t) = (\gamma_1'(t), \dots, \gamma_n'(t)) \neq \vec{0}$$

Terminology: (at $\gamma(t)$)

tangent vector $\gamma'(t)$

unit tangent vector $T(t) := \dfrac{\gamma'(t)}{\|\gamma'(t)\|}$

E.g.) Circle in $\mathbb{R}^2$



$$\gamma(t) = (R\cos t, R\sin t)$$

$$\gamma'(t) = (-R\sin t, R\cos t)$$

$$T(t) = (-\sin t, \cos t)$$

Q: How to do integration on a curve in $\mathbb{R}^n$?

To integrate a function $f : \mathbb{R}^n \to \mathbb{R}$ along a parametrized curve $\gamma : [a,b] \to \mathbb{R}^n$, we simply define:

$$\boxed{\int_\gamma f\, ds := \int_a^b f(\gamma(t)) \cdot |\gamma'(t)|\, dt}$$

Remark: By the Change of Variables Thm, the integral $\int_\gamma f\, ds$ is determined by f and the image C of the curve γ but NOT depend on how the curve is parametrized. Notation: $\int_C f\, ds$

E.g.) The two parametrizations

$$\gamma_1(t) = (\cos t, \sin t), \quad t \in [0, 2\pi]$$

$$\gamma_2(u) = (\cos 2u, \sin 2u), \quad u \in [0, \pi]$$

both parametrize the same unit circle in $\mathbb{R}^2$.

$$\int_{\gamma_1} 1\, ds = \int_0^{2\pi} 1 \cdot |\gamma_1'(t)|\, dt = 2\pi$$

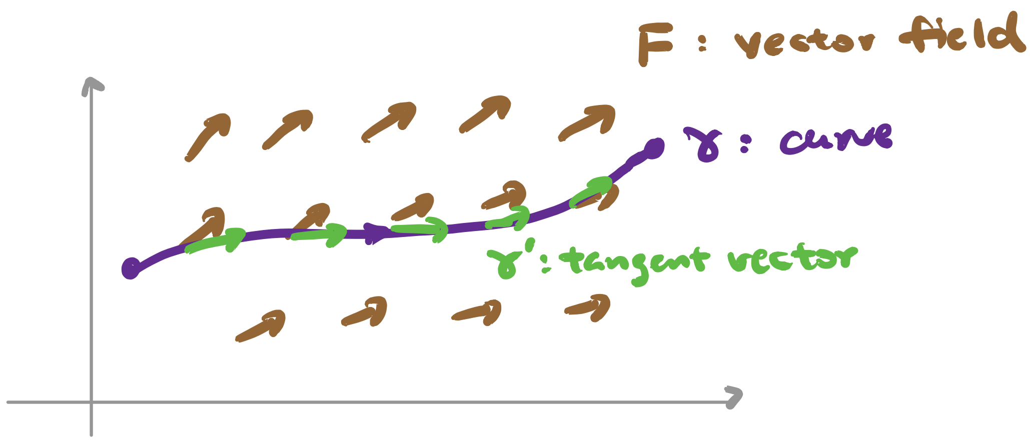
$$\|$$

$$\int_{\gamma_2} 1\, ds = \int_0^{\pi} 1 \cdot |\gamma_2'(u)|\, du = 2\pi$$

Def$^{\underline{n}}$: The length of a parametrized curve $\gamma: [a,b] \to \mathbb{R}^n$ is defined to be $\int_\gamma 1\, ds$.

Besides functions, sometimes we want to integrate more general objects along a curve in $\mathbb{R}^n$. In particular, given a vector field $F: \mathbb{R}^n \to \mathbb{R}^n$ and a parametrized curve $\gamma: [a,b] \to \mathbb{R}^n$, we define the line integral of F along γ as

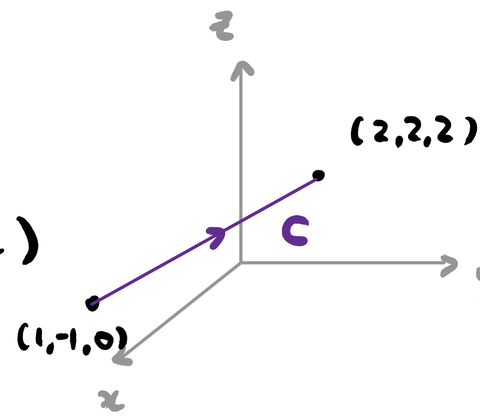
$$\boxed{\int_\gamma F \cdot d\vec{r} := \int_a^b F(\gamma(t)) \cdot \gamma'(t)\, dt}$$



Remark: Physically, the integral above computes the "work done" by a force F in displacing an object along the path γ

Example 1 : Evaluate the line integral of the vector field $F(x,y,z) = (0, 0, xy)$ along the line segment C from $(1,-1,0)$ to $(2,2,2)$.

Solution:



Step 1 : Parametrize the curve.

$$\gamma(t) = (1-t)(1,-1,0) + t\,(2,2,2)$$

$$= (1+t,\ -1+3t,\ 2t)$$

where $0 \le t \le 1$.

Step 2 : Evaluate the line integral.

$$F(\gamma(t)) = (0,\ 0,\ (1+t)(-1+3t))$$

$$\gamma'(t) = (1,3,2)$$

$$F(\gamma(t))\cdot\gamma'(t) = 2(1+t)(-1+3t)$$

$$\int_C F\cdot d\vec{r} = \int_0^1 F(\gamma(t))\cdot\gamma'(t)\,dt = \int_0^1 2(1+t)(-1+3t)\,dt$$

$$= 2\int_0^1 (-1+2t+3t^2)\,dt = 2 \; *$$

$\blacksquare$

We can write the line integrable as the integral of the function $F \cdot T$ along the same curve:

$$\int_\gamma F \cdot d\vec{r} = \int_a^b F(\gamma(t)) \cdot \boxed{\frac{\gamma'(t)}{\|\gamma'(t)\|}} \cdot \|\gamma'(t)\| \, dt$$

$$= \int_\gamma F \cdot \boxed{T} \, ds$$

This implies that $\int_\gamma F \cdot d\vec{r}$ remains unchanged if we re-parametrize the curve in the same SAME orientation.

Notation: $\int_C F \cdot d\vec{r}$

Prop:
$$\boxed{\int_{-C} F \cdot d\vec{r} = -\int_C F \cdot d\vec{r}}$$

where $-C$ denotes the curve C with reversed orientation.

Proof: Suppose C is parametrized by

$$\gamma(t) : [a,b] \to \mathbb{R}^n$$

then $-C$ is parametrized by

$$\bar{\gamma}(u) := \gamma(a+b-u) : [a,b] \to \mathbb{R}^n$$

Therefore, we have

$$\int_{-C} F \cdot d\vec{r} = \int_a^b F(\bar{\gamma}(u)) \cdot \bar{\gamma}'(u)\, du$$

$$= \int_a^b F(\gamma(a+b-u)) \cdot [-\gamma'(a+b-u)]\, du$$

$$\begin{pmatrix} \text{Substitute} \\ t = a+b-u \end{pmatrix} = -\int_a^b F(\gamma(t)) \cdot \gamma'(t)\, dt = -\int_C F \cdot d\vec{r}$$

$\blacksquare$

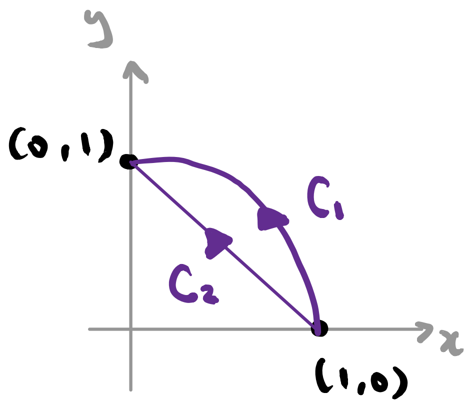
Exercise: Re-do Example 1 for the same curve with reversed orientation.

The following example shows that the line integral, in general, depends not just on the endpoints of the path, but also on the particular path joining them.

Example 2: Evaluate the line integral of the vector field $F(x,y) = (-y, x)$ along each of the parametrized curves C_1 and C_2 joining $(1,0)$ to $(0,1)$ given by:

$C_1: \quad \gamma_1(t) = (\cos t, \sin t), \quad 0 \le t \le \frac{\pi}{2}$

$C_2: \quad \gamma_2(t) = (1-t, t), \quad 0 \le t \le 1$



Solution:

Along C_1, we have

$$\gamma_1(t) = (\cos t, \sin t)$$

$$\gamma_1'(t) = (-\sin t, \cos t)$$

$$F(\gamma_1(t)) = (-\sin t, \cos t)$$

$$F(\gamma_1(t)) \cdot \gamma_1'(t) = \sin^2 t + \cos^2 t = 1$$

$$\Rightarrow \int_{C_1} F \cdot d\vec{r} = \int_0^{\frac{\pi}{2}} 1 \cdot dt = \frac{\pi}{2} \quad *$$

Along C_1, we have

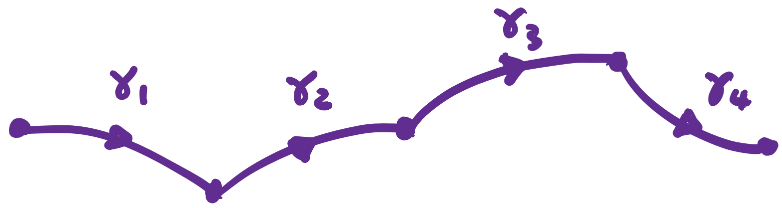
$$\gamma_2(t) = (1-t, t), \quad F(\gamma_2(t)) = (-t, 1-t)$$

$$\gamma_2'(t) = (-1, 1), \quad F(\gamma_2(t)) \cdot \gamma_2'(t) = 1$$

$$\Rightarrow \int_{C_2} F \cdot d\vec{r} = \int_0^1 1 \cdot dt = 1 \quad *$$

Note: $\int_{C_1} F \cdot d\vec{r} \neq \int_{C_2} F \cdot d\vec{r}$ even though C_1, C_2 have the SAME endpoints.

Remark: All the discussions above can be done on "piecewise" C^1 curves like this:



by breaking up the integrals and evaluate on each C^1 curve γ_i and sum them up.